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Homogeneous Interference Game in Wireless Networks

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Abstract

In many modern wireless scenarios, various stations contend for transmission in a an interference-bound environment. The main concern is the ability of the receiving stations to distinguish between the signal of the message and the noise created by other signals in the same proximity. We concentrate on a wireless environment where fixed stations need to communicate with clients in their vicinity, over a common communication channel, and focus on a distributed setting where there is no central entity managing the various transmissions. In such systems, unlike other multiple access environments, several transmissions may succeed simultaneously, depending on spatial interferences between the different stations.

We use a game theoretic view to model the problem, where the stations are selfish agents which aim at maximizing their success probability. We show that when interferences are homogeneous, the system's performance necessarily suffers an exponential degradation of performance in an equilibrium, due to the selfishness of the stations. However, when using a proper penalization scheme for aggressive stations, we can ensure the system's performance value is at least $1/e$ from the optimal value, while still being at equilibrium.

1 Introduction

Wireless networks often involve the joint usage of common communication channels in a multiple access environment. In most of the models capturing such settings, simultaneous transmission by more than one station results in a collision causing all transmissions at that time to fail. Methods like collision detection (CSMA/CD) and collision avoidance (CSMA/CA), as well as other related protocols such as the various variants of the Aloha protocol, are used in such scenarios in order to deal with collision, and to maximize the system's throughput. In many current wireless networks, such as mesh WiFi networks, or 802.15 clusters, simultaneous usage of the same wireless channel is possible. Consider for example the settings described in Figure 1, where we outline two stations, A, B and their transmission ranges. If the clients of A and B are a and b respectively, then simultaneous transmissions will cause a collision at client a, while b can receive the message from B. However, if the clients of A and B are a' and b respectively, then simultaneous transmissions will both succeed, since they do not collide at either of the receiving ends.

In wireless networks where channel access need not be exclusive, one of the major optimization issues is the efficient use of radio resources. For example, in currently deployed wireless mesh networks, the currently available MAC protocols, as well as routing protocols, do not provide sufficient scalability and consequently the throughput tends to drop significantly as the number of nodes increases [1].

In this paper we consider the problem of joint usage of a common communication channel by a finite number of stations, where stations are always backlogged, i.e., always have a packet to send. We present a generalization of classic multiple access models by introducing the notion of *spatial interference parameters*, which capture the pairwise interferences between the stations contending for the common radio resource.[1] It is important to notice that in this model several transmissions may succeed simultaneously, and thus the commonly assumed upper bound of one on the overall throughput of the system no longer holds. The overall number of successful transmissions at any time

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[1] We do not consider any higher-order interferences. We note that our model, although focusing solely on pairwise-interferences, already generalizes standard multiple access models commonly used in the analysis of such networking environments.

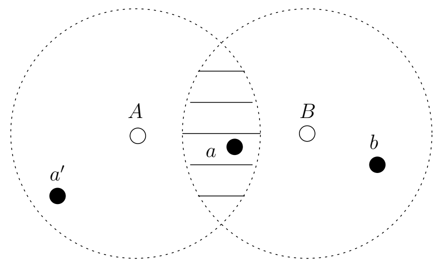

Figure 1: Outline of two stations, A, B and their transmissions ranges.

can take any value between 0 and n, where n is the number of stations in the network. The exact value depends on the inter-station interferences.

As a preliminary step in understanding this model, we focus our attention on the case of homogeneous interferences, where every station inflicts the same amount of interference on any other station.

Note that is we adopt the geometric interpretation of Figure 1, in the plane we can have at most 3 agents, all inflicting the same amount of interference on each other, whereas for n agents we need $n-1$ dimensions. Indeed, such homogeneous interferences will usually not provide an accurate modeling for real-life scenarios where different agents have different interference patterns. Yet, we believe that a better understanding of the restricted settings (where the system is described by a single parameter) is both interesting and serves as an important step toward providing insights into understanding more general non-homogeneous environments.

We conduct an analytical investigation of the performance of the network from a game-theoretic point of view, where we assume stations are selfish agents and each such agent aims at maximizing its utility.

To evaluate the throughput of the system, we use the notions of *price of anarchy* [14] and *price of stability* [5, 9], which compare the system's performance in Nash equilibrium, (where no agent can do better by unilaterally altering its chosen strategy), to the optimal throughput of the system.

We show that if the agent's only goal is maximizing their success probability then the system's performance necessarily suffers an exponential degradation of performance in an equilibrium. However, when agents consider also other parameters, such as for example their power consumption, we can ensure the system's performance value at equilibrium is at least $1/e$ from the optimal value. This is done by defining a proper penalization scheme for aggressive stations that is proportional to the power used by this agent.

Note that our proposed model is general, and can be applied to many wireless environments, such as the ones emerging in wireless mesh networks, wireless personal area networks (WPAN), and other ad-hoc networking environments, where taking spatial considerations into account may significantly increase the network's throughput and efficiency.

In the remainder of this section we give a formal definition of the underlying model of interferences, present a summary of our results, and discuss related work. In the sequel we present our analytical results for homogeneous interferences, and conclude with some open questions.

1.1 Model

We model our problem as a game played by selfish agents. We consider a system consisting of n agents using a common wireless medium. For every agent i, we let $S = [0, 1]$ be the strategy space of agent i, and let $R_i \in S$ denote a strategy chosen by agent i. We refer to R_i as the *probability* that agent i transmits. R_i can also be considered as the *rate* in which agent i transmits. Due to interferences, the probability of a *successful* transmission also depends upon the transmission of other agents, or alternatively, the *effective rate* an agent eventually experiences depends upon the transmission rates of the other agents. Given a profile $\overline{R} = (R_1, \ldots, R_n) \in [0, 1]^n$, we define the *success probability* of agent i's transmission (or alternatively, the *effective rate* obtained by agent i) as:

$$r_i(\overline{R}) = R_i \cdot \Pi_{j \neq i}(1 - \alpha_{i,j} R_j),$$

where for every $1 \leq i, j \leq n$, $\alpha_{i,j} \in [0,1]$ is a fixed network-dependent parameter denoting the amount of interference inflicted on i upon simultaneous transmission of both i and j. In what follows we will usually refer to the agents' strategies as transmission probabilities.

One way to think about the $\alpha_{i,j}$s is considering them as the probability that a transmission by j will interfere with a transmission of i. Then in order for i's transmission to succeed, none of the transmission of the other station should interfere. Clearly, if for all i, j, $\alpha_{i,j} = 0$, i.e., there are no interferences, then in any reasonable setting the selfish behavior of the agents will result in all agents transmitting with probability 1, which implies an optimal use of resources. On the other hand, if for all i, j, $\alpha_{i,j} = 1$, then our model coincides with classic multiple access models, where in any case of simultaneous transmissions a collision occurs, resulting in the failure of all the transmissions. We refer to the interferences as *homogeneous* if there exists some $\alpha \in (0,1)$ such that for all i, j, $\alpha_{i,j} = \alpha$. By the above observations, when considering homogeneous interferences, we restrict our attention to the case where for all i, j, $\alpha_{i,j} = \alpha \in (0,1)$.

Given a profile $\overline{R} = (R_1, \ldots, R_n)$, the *social welfare* $\varphi(\overline{R})$ is considered to be the overall use of resources in the system, i.e.

$$\varphi(\overline{R}) = \sum_{i=1}^{n} r_i(\overline{R}) = \sum_{i=1}^{n} R_i \prod_{j \neq i} (1 - \alpha_{i,j} R_j).$$

$\varphi(\overline{R})$ can be interpreted as the expected number of successful transmission, or alternatively, as the overall effective usage of bandwidth in the network. We refer to $\varphi(\overline{R})$ as the *throughput* of the system. Note that a-priori, $\varphi(\overline{R})$ can take any value between 0 and n, where the former is its value e.g. in case where $\alpha_{i,j} = 1$ for all i, j, and $R_i = 1$ for all i, and the latter is its value where $\alpha_{i,j} = 0$ for all i, j, and $R_i = 1$ for all i. In what follows we refer to a profile $\overline{R}$ as *uniform*, if $R_i = R_j$ for all i, j.

For every agent i, we let $U_i(\overline{R})$ be the utility function of agent i, assuming agents play profile $\overline{R}$. In the following sections we consider several choices for these utility functions, and discuss the system's performance where agents are selfish, and aim at maximizing their own utility, regardless of the effect their choices have on the overall social welfare. We refer to the above setting as the *homogeneous interferences multiple-access (HIMA) game*.

Given any profile $\overline{R} = (R_1, \ldots, R_n)$, we let $\overline{R}_{-i}$ denote the subprofile defined by strategies of all agents except for agent i. We further denote by $(\overline{R}_{-i}, R'_i)$ the profile where every agent other than i plays the same strategy as in $\overline{R}$, while agent i plays strategy R'_i. A profile $\overline{R}$ is said to be a *Nash equilibrium* if for every i, and every $R'_i \in [0,1]$, $U_i(\overline{R}) \geq U_i(\overline{R}_{-i}, R'_i)$. Intuitively, a profile is at Nash equilibrium if no agent can increase its benefit by unilaterally deviating from his choice. Given any $n \in \mathbb{N}$, we use $\overline{R}^{(n)}_{\text{NE}}$ to denote a Nash equilibrium profile for n agents, and use $\overline{R}^{(n)}_{\text{OPT}}$ to denote any profile for n agents which maximizes the social welfare. Assuming a Nash equilibrium exists, we use the notion of *Price of Anarchy (PoA)* in order to evaluate this effect, defined by the supremum over all Nash equilibria $\overline{R}^{(n)}_{\text{NE}}$ of the ratio between $\varphi(\overline{R}^{(n)}_{\text{OPT}})$ and $\varphi(\overline{R}^{(n)}_{\text{NE}})$, capturing the performance of the worst case equilibrium. We further consider the notion of *Price of Stability (PoS)*, defined by the infimum of the above ratio over all Nash equilibria, capturing the performance of the best case equilibrium.

1.2 Our Results

We study the rational choices of agents in an HIMA game, and analyze the performance of Nash equilibria compared to the optimal performance. We focus on the case of homogeneous interferences, and show that when the utility of an agent is its effective rate, then selfishness can cause the system's performance to be up to an exponential factor away from the optimal performance. Specifically, we show that for any constant α, the price of anarchy as well as the price of stability are exponential in the number of agents, i.e., *any* equilibrium suffers an exponential degradation in performance. These results appear in Section 2.

We then turn to explore the effect of penalization, and to what extent does such an approach provide better system performance at a state of equilibrium. We show that there exists a penalty function which is proportional to the amount of aggressiveness demonstrated by an agent, such that for the case where the utility of an agent is the sum of its rate and its penalty, then the price of stability with regards to the resulting *coordinated equilibria* can be made to drop to at most $e \approx 2.718$, thus demonstrating that an exponential improvement is possible compared to the uncoordinated case. We further show that for interferences which are not too large, namely, for $\alpha \leq 2/e \approx 0.735$, the price of anarchy is also bounded by e, thus ensuring that the degradation in performance due to the selfishness of the agents can be guaranteed to be made very small. These results mean that if we impose these penalties upon the agents, either in the form of payment for transmission to the network operator, or considering them as an intrinsic cost suffered by the agent due to transmission (e.g., due to power consumption), then the performance can be dramatically improved compared to the general case where the agent's utility is merely its effective rate. These results are presented in Section 3. We note that our results for the homogeneous settings also extend to the finite horizon repeated HIMA game [22].

1.3 Previous Work

Issues involving selfish behavior of agents in multiple access environments have received much attention in recent years. The slotted Aloha model was studied in, e.g., [2, 4, 13, 16–18, 26], in the Markovian setting, presenting conditions on the system's stability, and investigating convergence to equilibrium. Other models of interferences in wireless networks in Markovian settings are discussed in [7, 12, 23].

Additional works considered the issue of rate control in wireless networks, and discussed several game theoretical model of such games [20, 21]. Some recent work [19, 26], has also considered the role of introducing costs for transmissions and its effect on the stability of the system. Other aspects of selfish behavior in CSMA/CA networks, such as the effect of the selfish deviation of agents from a protocol, were studied in [25]. Power control penalization schemes for wireless networks were discussed in [24, 27], whereas access pricing in wireless mesh networks is discussed in [15].

There has been much interest in game theoretic perspectives of combinatorial optimization problems, most notably following the introduction of the notion of price of anarchy in [14]. Recent works have considered the role of taxation, or penalties, on the performance of non-cooperative systems with selfish agents, thus resulting in *coordinated Nash equilibria*. Most of these works involve setting tolls in order to minimize congestion [6, 8, 11].

Correlated Nash equilibrium in multiple access environments is discussed in [3], where an arbitrator can send some random signal to every agent. The signal can be used by the agent in choosing its strategy, such that the overall system performance improves by agents using the additional information provided by the arbitrator, thus introducing some level of coordination.

Another recent study focused on the time it takes all agents to successfully transmit one packet each [10], and every agent's goal is to minimize its delay. They show that while the price of anarchy is exponential (w.h.p.), the price of stability is bounded by a constant (w.h.p.).

2 General Nash Equilibria

In this section we present several analytical results as to the effect of selfishness upon the performance of the network, in the theoretical case where interferences are homogeneous, i.e., for every i, j, $\alpha_{i,j} = \alpha$, for some system's parameter $\alpha \in (0, 1)$. We first consider the simple utility function $U_i(\overline{R}) = r_i(\overline{R})$, and show that in such a case, the system's performance can be very far from optimal. Specifically we prove the following theorem:

Theorem 2.1. *Given n stations, and any $k \in \{1, \dots, n-1\}$,*

1. *If $\alpha \in \left[\frac{1}{k+1}, \frac{1}{k}\right)$ then*
$$\mathrm{PoA}^{(n)} = \mathrm{PoS}^{(n)} = \frac{k}{n(1-\alpha)^{n-k}}.$$
2. *If $\alpha \leq \frac{1}{n}$ then $\mathrm{PoA}^{(n)} = 1$.*

Note that Theorem 2.1 implies that for any constant $m \in \mathbb{N}$, for $\alpha = 1/m$ we have

$$\mathrm{PoA}^{(n)} = \mathrm{PoS}^{(n)} = 2^{\Omega(n)}.$$

In what follows we provide the necessary elements in order to prove the above theorem. The following lemma is straight forward and follows immediately from the definition of the utility function:

Lemma 2.2. *For utility functions $U_i(\overline{R}) = r_i(\overline{R})$, the only Nash equilibrium solution is obtained by the uniform profile $\overline{R}$ where every station i plays the strategy $R_i = 1$. The social welfare value of this Nash equilibrium is $n(1-\alpha)^{n-1}$.*

Lemma 2.2 implies in particular that since there is only one Nash equilibrium in these settings, then the price of stability equals the price of anarchy. In order to determine this value, we now turn to analyze the value of a profile which maximizes the social welfare. We first analyze the value of the social welfare function on the boundary of the profiles domain $[0,1]^n$, and then turn to analyze the maximum value obtained in the interior of the domain.

Since φ is symmetric, any two integral profiles $\overline{R}, \overline{R}' \in \{0,1\}^n$ having the same number of 1's, satisfy $\varphi(\overline{R}) = \varphi(\overline{R}')$. Let $B_k = (R_1, \ldots, R_n)$ denote any profile with exactly k 1's. It therefore follows that the value in every extreme point where k stations play the 1-strategy and $n-k$ stations play the 0-strategy, is given by

$$\begin{aligned} v_k &= \varphi(B_k) \\ &= \textstyle\sum_{i:R_i=1} R_i \prod_{j\neq i}(1-\alpha R_j) \\ &\quad + \textstyle\sum_{i:R_i=0} R_i \prod_{j\neq i}(1-\alpha R_j) \\ &= \textstyle\sum_{i:R_i=1} \prod_{j\neq i, R_j=1}(1-\alpha) \\ &= k(1-\alpha)^{k-1}. \end{aligned}$$

The following lemma shows an ordering of the values of v_k, depending on the value of α.

Lemma 2.3. *$v_k > v_{k-1}$ if and only if $\alpha < \frac{1}{k}$.*

Proof. Requiring that $v_k > v_{k-1}$ is equivalent to requiring $v_k / v_{k-1} > 1$. By the definition of v_k, this holds if and only if

$$\frac{k(1-\alpha)^{k-1}}{(k-1)(1-\alpha)^{k-2}} > 1,$$

which by further simplification results in $1 - \alpha > \frac{k-1}{k}$. This indeed holds if and only if $\alpha < \frac{1}{k}$. □

The following is an immediate corollary of Lemma 2.3:

Corollary 2.4. *If $\alpha \in [\frac{1}{k+1}, \frac{1}{k})$ then $\max_j v_j = v_k$.*

Figure 2 outlines the different values of v_k, as a function of α. Recall that v_k is the social value when exactly k stations transmit, and all other $n-k$ stations refrain from transmitting.

Since clearly $\varphi(\overline{R}^{(n)}_{\text{OPT}}) \geq v_k$ for all k and for all α, we therefore have $\varphi(\overline{R}^{(n)}_{\text{OPT}}) \geq \max_k v_k$ for all α. In order to show that indeed $\varphi(\overline{R}^{(n)}_{\text{OPT}}) = \max_k v_k$, we wish to show that the maximum of $\varphi(\cdot)$ is not obtained in the interior of the domain. We first show that there is only one possible extreme point x_0 in the interior of the domain $(0,1)^n$. We further show that $\varphi(x_0) \leq \max_k v_k$, which therefore implies that $\varphi(\overline{R}^{(n)}_{\text{OPT}}) = \max_k v_k$ for all α.

Lemma 2.5. *In the n-stations HIMA game, given any $\alpha \in (0,1)$, the only possible extreme point of the social welfare function $\varphi(\cdot)$ in the interior of $(0,1)^n$ is $\overline{R}_0 = (\frac{1}{\alpha n}, \ldots, \frac{1}{\alpha n})$.*

Proof. Note that by algebraic simplification we have

$$\begin{aligned} \varphi(\overline{R}) &= \textstyle\sum_{i=1}^{n} R_i \prod_{j\neq i}(1-\alpha R_j) \\ &= R_n \textstyle\prod_{j\neq n}(1-\alpha R_j) \\ &\quad + \textstyle\sum_{i\neq n} R_i \prod_{j\neq i}(1-\alpha R_j) \\ &= R_n \textstyle\prod_{j\neq n}(1-\alpha R_j) \\ &\quad + (1-\alpha R_n) \textstyle\sum_{i\neq n} R_i \prod_{j\neq i,n}(1-\alpha R_j) \\ &= R_n \Big(\textstyle\prod_{j\neq n}(1-\alpha R_j) \\ &\quad -\alpha \textstyle\sum_{i\neq n} R_i \prod_{j\neq i,n}(1-\alpha R_j)\Big) \\ &\quad + \textstyle\sum_{i\neq n} R_i \prod_{j\neq i,n}(1-\alpha R_j). \end{aligned}$$

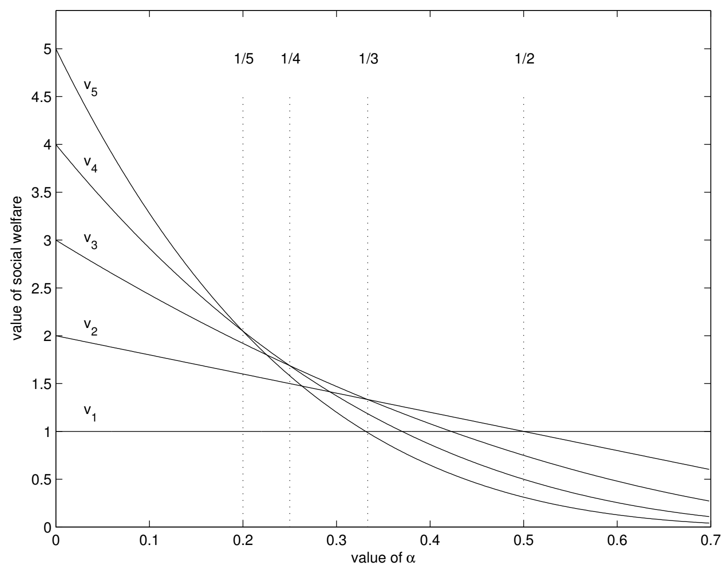

Figure 2: Different values of v_k, depending on the value of α, up to $k = 5$.

By taking derivatives we obtain

$$\frac{\partial \varphi}{\partial R_n} = \prod_{j \neq n} (1 - \alpha R_j) - \alpha \sum_{i \neq n} R_i \prod_{j \neq i,n} (1 - \alpha R_j),$$

where verifying when the above is zero and reordering we obtain (since $\alpha \in (0, 1)$)

$$0 = 1 - \alpha \sum_{i \neq n} \frac{R_i}{1 - \alpha R_i},$$

which in turn yields

$$\frac{\alpha R_1}{1 - \alpha R_1} = 1 - \alpha \sum_{i \neq 1,n} \frac{R_i}{1 - \alpha R_i}.$$

Note that the term on the right hand side is independent of R_1 and R_n. Following the same above procedure of taking first the derivative with regards to R_1, and then isolating R_n results in

$$\frac{\alpha R_n}{1 - \alpha R_n} = 1 - \alpha \sum_{i \neq 1,n} \frac{R_i}{1 - \alpha R_i} = \frac{\alpha R_1}{1 - \alpha R_1}.$$

It therefore follows that $R_1 = R_n$. Since the above argument holds for any $1 \leq i < j \leq n$, we have that the only point where all derivatives are zero satisfies $R_i = R_j$ for all i, j.

We can therefore consider the possible extreme points of φ in the interior of the domain as a subset of the n-tuples of the extreme points of the single variable function ψ, where $\psi(x) = nx(1 - \alpha x)^{n-1}$.

Taking derivatives we obtain

$$\begin{aligned} \psi'(x) &= n(1 - \alpha x)^{n-1} \\ &\quad + nx(n-1)(1 - \alpha x)^{n-2}(-\alpha) \\ &= [(1 - \alpha x) - \alpha(n-1)x]\, n(1 - \alpha x)^{n-2} = 0. \end{aligned}$$

There are two options to consider. First consider the case where $(1 - \alpha x) = 0$ which implies $x = \frac{1}{\alpha}$. This is impossible in the interior of the domain since $\alpha < 1$. It therefore follows that

$$(1 - \alpha x) - \alpha(n-1)x = 0.$$

It therefore follows that $x_0 = \frac{1}{\alpha n}$, which in turn implies that the only possible extreme point of $\varphi(\cdot)$ in the interior of $(0,1)^n$ is $\overline{R}_0 = (\frac{1}{\alpha n}, \ldots, \frac{1}{\alpha n})$. □

The following lemma suffices in order to show that $\varphi(\overline{R}_0) \leq \max_j v_j$.

Lemma 2.6. *Consider the profile $\overline{R}_0 = (\frac{1}{\alpha n}, \ldots, \frac{1}{\alpha n})$. For any $k = 1, \ldots, n-1$, if $\alpha \in [\frac{1}{k+1}, \frac{1}{k})$, then $\varphi(\overline{R}_0) \leq v_k$.*

Proof. By definition, we need to show

$$\varphi(\overline{R}_0) = \frac{(n-1)^{n-1}}{\alpha \cdot n^{n-1}} \leq k(1-\alpha)^{k-1} = v_k,$$

or equivalently, we need to show that for any $\alpha \in [\frac{1}{k+1}, \frac{1}{k})$

$$f_k(\alpha) = k \cdot n^{n-1} \cdot \alpha(1-\alpha)^{k-1} - (n-1)^{n-1} \geq 0.$$

Taking derivatives, we obtain

$$\begin{aligned} f'_k(\alpha) &= k \cdot n^{n-1}(1-\alpha)^{k-1} \\ &\quad - k \cdot n^{n-1} \cdot \alpha(k-1)(1-\alpha)^{k-2} \\ &= k \cdot n^{n-1}(1-\alpha)^{k-2}\left[(1-\alpha) - (k-1)\alpha\right] \\ &= k \cdot n^{n-1}(1-\alpha)^{k-2}\left[1 - k \cdot \alpha\right]. \end{aligned}$$

The derivative is zero in one of two cases:

1. $\alpha = 1$: This is out of bound.
2. $\alpha = \frac{1}{k}$.

It follows that for $\alpha < \frac{1}{k}$, $f_k(\alpha)$ is strictly monotone increasing. It therefore suffices to show that $f_k(\frac{1}{k+1}) \geq 0$, i.e.:

$$\begin{aligned} f_k(\tfrac{1}{k+1}) &= k \cdot n^{n-1} \cdot \tfrac{1}{k+1} \cdot (1 - \tfrac{1}{k+1})^{k-1} \\ &\quad -(n-1)^{n-1} \\ &= n^{n-1} \cdot \tfrac{k}{k+1} \cdot (1 - \tfrac{1}{k+1})^{k-1} \\ &\quad -(n-1)^{n-1} \\ &= n^{n-1} \cdot \tfrac{k}{k+1} \cdot (\tfrac{k}{k+1})^{k-1} \\ &\quad -(n-1)^{n-1} \\ &= n^{n-1}(\tfrac{k}{k+1})^k - (n-1)^{n-1} \\ &= n^{n-1}(1 - \tfrac{1}{k+1})^k - (n-1)^{n-1} \geq 0, \end{aligned}$$

which follows since,

$$(1 - \frac{1}{k+1})^k \geq (1 - \frac{1}{n})^{n-1},$$

for every $k = 1, \ldots n-1$, thus completing the proof.[2] □

The above lemma combined with Corollary 2.4 immediately implies the following corollary:

Corollary 2.7. *Given n agents, if $\alpha \in [\frac{1}{k+1}, \frac{1}{k})$, then $\varphi(\overline{R}^{(n)}_{\text{OPT}}) = v_k$.*

[2] This is due to the fact that the function $g(x) = (1 - 1/x)^{x-1}$ is monotone decreasing.

Figure 2 depicts the various plots of v_k as a function of α, up to $k = 5$. The maximal overall throughput as a function of α is given by the maximal curve among all v_ks. One can see that the overall throughput, which is also the optimal social welfare, increases as α approaches 0.

Combining Lemma 2.2 which shows that there exists a single Nash equilibrium solution $\overline{R}^{(n)}_{\text{NE}} = (1, \ldots, 1)$ whose value is $\varphi(\overline{R}^{(n)}_{\text{NE}}) = n(1-\alpha)^{n-1}$, along with Corollary 2.7, we can conclude the proof of Theorem 2.1.

As a consequence of Theorem 2.1, we restrict our attention in the following sections to the case where $\alpha \in (1/n, 1)$, since for $\alpha \leq 1/n$, the single Nash equilibrium of the HIMA game is indeed optimal. The following sections present a penalization scheme which enables the system to obtain a much better throughput, while still being at equilibrium.

We note that the above results as to the price of anarchy of the homogeneous HIMA game are based on the fact that there exists a single Nash equilibrium. When considering the finite horizon repeated HIMA game, it is easy to verify that this repeated game is subgame perfect, since the strategy chosen by each player at the single game Nash equilibrium is a best response regardless of the strategies chosen by the other players. It therefore follows that our analysis implies the same results for the subgame perfect equilibria emanating in the finite horizon repeated HIMA game [22].

3 Coordinated Nash Equilibria

In this section we introduce a penalty based scheme, where every station i incurs a penalty $p_i(\cdot)$ for transmission. We consider two types of penalties. The first type depends upon the choices of all the stations in the system, i.e., $p_i(\overline{R})$, while the second type only depends upon the choice of station i, i.e., $p_i(R_i)$. We refer to the former as an *exogenous penalty*, whereas the latter is referred to as an *endogenous penalty*. The general form of the utility function of station i is therefore $U_i(\overline{R}) = r_i - p_i$ (see [27] for a similar approach in the context of power control in cellular networks). We use the notion of coordinated Nash equilibrium and show that for both penalty functions, the selfishness of the stations does not result in more than a constant factor degradation in performance compared to the optimal performance. This should be contrasted with the results presented in the previous section showing that the price of stability for the uncoordinated case can be exponential in the number of stations.

We first show that there exists some $q_0 \in [0, 1]$ such that the uniform profile $\overline{R}$ where $R_i = q_0$ for all i implies a mere constant degradation in performance compared to the optimal throughput possible. Note however that such a uniform profile need not be at Nash equilibria. We then show that there exist penalty functions which cause such a uniform profile to be at Nash equilibrium. It therefore follows that by the use of appropriate penalties, selfishness can be tamed into providing a throughput that is at most a constant factor far from the optimal throughput.

3.1 The Power of Uniform Profiles

Given any $q \in [0, 1]$, let $\overline{R}^q$ denote the uniform profile where $R_i = q$ for all i. Note that the social welfare value of any such profile $\overline{R}^q$ is given by the function

$$\psi(q) = nq(1-\alpha q)^{n-1}.$$

As shown in the proof of Lemma 2.5, the value of q which maximizes $\psi(\cdot)$ is $q_0 = \frac{1}{\alpha n}$. It follows that the social welfare value of the profile $\overline{R}^{q_0}$ is

$$\psi(q_0) = \frac{1}{\alpha}\left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{1}{e\alpha},$$

where the inequality follows from the fact that $\left(1 - \frac{1}{n}\right)^{n-1}$ is strictly monotone decreasing, and converges to e^{-1}.

As we have seen, the optimal value of the social welfare function for $\alpha \in [\frac{1}{k+1}, \frac{1}{k})$ [3] is obtained for a profile where k stations play the 1-strategy, and $n - k$ stations play the 0-strategy, resulting in a social welfare value of

$$\varphi(\overline{R}^{(n)}_{\text{OPT}}) = k(1-\alpha)^{k-1} < k\left(1 - \frac{1}{k}\right)^{k-1} \leq \frac{1}{\alpha}.$$

[3] Equivalently, $k < \frac{1}{\alpha} \leq k + 1$.

It follows that

$$\frac{\varphi(\overline{R}^{(n)}_{\text{OPT}})}{\varphi(\overline{R}^{q_0})} \leq e.$$[4]

In the following sections we indeed show that we can choose a penalty function such that the profile $\overline{R}^{q_0}$ is at Nash equilibrium.

3.2 Exogenous Penalties

Let $q \in [0, 1]$, and consider the following utility function:

$$U_i^q(\overline{R}) = R_i \prod_{j \neq i} (1 - \alpha R_j)(2q - R_i),$$

which can be cast as a utility function of the form

$$U_i^q(\overline{R}) = r_i(\overline{R}) - p_i^q(\overline{R}),$$

where the exogenous penalty is defined by

$$p_i^q(\overline{R}) = \prod_{j \neq i} (1 - \alpha R_j)((1 - 2q)R_i + R_i^2).$$

Assume all stations except for i play strategy q. It follows that

$$\begin{aligned} U_i^q(R) &= R_i(1 - \alpha q)^{n-1}(2q - R_i) \\ &= (1 - \alpha q)^{n-1}(2qR_i - R_i^2). \end{aligned}$$

By taking derivatives, we obtain that the maximum is obtained for $R_i = q$, i.e., the uniform profile $\overline{R}^q$ is at Nash equilibrium.

It therefore follows that the price of stability is at least

$$\max_q \frac{\varphi(\overline{R}^{(n)}_{\text{OPT}})}{\varphi(\overline{R}^q)}.$$

In addition, since choosing $R_i = q$ is the best response of station i regardless of the strategy chosen by any station $j \neq i$, we can conclude that the uniform profile $\overline{R}^q$ is the only Nash equilibrium solution, hence the price of anarchy is the same as the price of stability.

Combining this result with the result presented in the previous section, for $q_0 = \frac{1}{\alpha n}$, we obtain the following theorem:

Theorem 3.1. *For every station i there exists an exogenous penalty function $p_i(\overline{R})$ for which the price of anarchy, as well as the price of stability, are at most e.*

Although Theorem 3.1 guarantees that aggressiveness can be tamed, and it is possible to use exogenous penalties in order to obtain a better performance at equilibrium, this might not be completely satisfactory. Exogenous penalties incurred by a station might change even if the station does not change its strategy. This might not be considered a handicap if the penalties cannot increase if the station remains put, however in our case, other stations being less aggressive actually increases the penalty incurred by a station, even if this station does not change its strategy. We address this issue in the following section, and present an endogenous penalty scheme, in which the penalty imposed on a station depends solely on its strategy, where increased aggressiveness is matched by increased penalties.

[4] Note that for α sufficiently distant from 1, a much better bound can be obtained. E.g., already for $\alpha < \frac{1}{2}$ this analysis yields that the ratio is at most $e/2$.

3.3 Endogenous Penalties

In this section we use the insights obtained from the proof of Theorem 3.1 in order to present for every station i an endogenous penalty function p_i which is independent of the strategy adopted by any station other than i, i.e., the penalty function of station i depends only on R_i.

Specifically, given any $q \in [0,1]$, we consider for every station i the utility function

$$U_i^q(\overline{R}) = r_i(\overline{R}) - p_i^q(R_i),$$

where the endogenous penalty function is defined by

$$p_i^q(R_i) = (1-\alpha q)^{n-1}((1-2q)R_i + R_i^2).$$

Assume all stations but i play the strategy q. It follows that the utility function of station i is the same as in the previous section, i.e.,

$$\begin{aligned} U_i^q(R_i) &= R_i(1-\alpha q)^{n-1}(2q - R_i) \\ &= (1-\alpha q)^{n-1}(2qR_i - R_i^2), \end{aligned}$$

which in turn implies that the best strategy for station i to play is $R_i = q$, hence the uniform profile $\overline{R}^q$ is at Nash equilibrium. Similarly to Theorem 3.1, we thus obtain the following theorem:

Theorem 3.2. *For every station i there exists an endogenous penalty function $p_i(R_i)$ for which the overall price of stability is at most e.*

In what follows we analyze the conditions for which the uniform profile $\overline{R}^q$ is actually the *only* Nash equilibrium. In any case for which there is a single Nash equilibrium, the price of stability is the same as the price of anarchy, which provides a guarantee upon the worst case equilibrium. Specifically, we prove the following theorem:

Theorem 3.3. *If $\alpha \leq \frac{2}{e}$, then for every station i there exists an endogenous penalty function $p_i(R_i)$ for which the overall price of anarchy is at most e.*

Proof. By considering the derivative of the utility function of station i, we obtain that

$$\frac{\partial U_i^q}{\partial R_i} = \prod_{j \neq i}(1-\alpha R_j) - (1-\alpha q)^{n-1}(1-2q+2R_i) = 0,$$

which implies

$$R_i = \frac{\prod_{j\neq i}(1-\alpha R_j)}{2(1-\alpha q)^{n-1}} - (\frac{1}{2} - q).$$

By further isolating R_k for some $k \neq i$, we obtain

$$\begin{aligned} R_i &= (1-\alpha R_k)\tfrac{\prod_{j\neq i,k}(1-\alpha R_j)}{2(1-\alpha q)^{n-1}} - (\tfrac{1}{2} - q) \\ &= \underbrace{\left[\frac{\prod_{j\neq i,k}(1-\alpha R_j)}{2(1-\alpha q)^{n-1}} - (\frac{1}{2} - q)\right]}_{a_{i,k}} \\ &\quad - \underbrace{\left[\alpha \cdot \frac{\prod_{j\neq i,k}(1-\alpha R_j)}{2(1-\alpha q)^{n-1}}\right]}_{b_{i,k}} R_k. \end{aligned}$$

Note that $a_{i,k}$ and $b_{i,k}$ are independent of R_i and R_k, hence we can write the relation between them as

$$R_i = a_{i,k} - b_{i,k}R_k.$$

By applying the same arguments, starting with considering the derivative $\frac{\partial U_k^q}{\partial R_k}$, and then isolating R_i, we can obtain that in the point where both derivatives are zero we must also have

$$R_k = a_{i,k} - b_{i,k} R_i.$$

Solving this system of linear equations we obtain that any point in which both the derivative of $U_i^q(R_i)$ and of $U_k^q(R_k)$ is zero, assuming $b_{i,k} \neq 1$, we must have $R_i = R_k$.[5] Since this applies to any i, k, we can conclude that if for all i, k, $b_{i,k} \neq 1$, then we have $R_1 = R_2 = \ldots = R_n$.

Assume now that for some $i \neq k$, $b_{i,k} = 1$. In this case, we have

$$b_{i,k} = \alpha \cdot \frac{\prod_{j \neq i,k}(1 - \alpha R_j)}{2(1 - \alpha q)^{n-1}} = 1,$$

which in turn implies

$$\prod_{j \neq i,k}(1 - \alpha R_j) = \frac{2(1 - \alpha q)^{n-1}}{\alpha}.$$

By taking $q = \frac{1}{\alpha n}$, we can conclude that

$$\prod_{j \neq i,k}(1 - \alpha R_j) = \frac{2(1 - \frac{1}{n})^{n-1}}{\alpha} > \frac{2e^{-1}}{\alpha}.$$

Since $\prod_{j \neq i,k}(1 - \alpha R_j) \leq 1$, we thus have $\alpha > \frac{2}{e}$.

It follows that for $q = \frac{1}{\alpha n}$, if $\alpha \leq \frac{2}{e} \sim 0.736$, then for all $i \neq k$, $b_{i,k} \neq 1$. In such a case, we are guaranteed to have $R_i = R_k$ for all i, k.

Assume $\alpha \leq \frac{2}{e}$, (and $q = \frac{1}{\alpha n}$) which implies that any Nash equilibrium solution must satisfy $R_i = R_k$ for all i, k. It follows that we seek a Nash equilibrium solution where the utility function of every station i is given by

$$\tilde{U}_i^q(x) = x(1 - \alpha y)^{n-1} - x(1 - \alpha q)^{n-1}(1 - 2q + x).$$

where all other stations except for i play the same strategy y. In order for this to be in equilibrium, the following conditions must hold:

1. $(\tilde{U}_i^q)'(x) = 0$.

2. $x = y$, since in any equilibrium all stations play the same strategy.

In order to see condition (1), consider

$$(\tilde{U}_i^q)'(x) = (1 - \alpha y)^{n-1} - (1 - \alpha q)^{n-1}(1 - 2q + 2x) = 0.$$

Since by condition (2) we must have $x = y$, it follows that we wish to find the solutions to the equation

$$\underbrace{(1 - \alpha y)^{n-1}}_{p(y)} - \underbrace{(1 - \alpha q)^{n-1}(1 - 2q + 2y)}_{\ell(y)} = 0.$$

Clearly, $y = q$ is a solution to this equation. Furthermore, this is the only solution to this equation. To see this note that $p(y)$ is strictly decreasing as a function of y, while $\ell(y)$ is strictly increasing as a function of y. It follows that any deviation from $y = q$ renders this equation false. It follows that $y = q$ is the only solution to this equation, hence the only Nash equilibrium solution is obtained for the uniform profile $\overline{R} = (\frac{1}{\alpha n}, \ldots, \frac{1}{\alpha n})$. □

[5] If we let $a = a_{i,k}$ and $b = b_{i,k}$, by substitution we have $R_i = a - b(a - bR_i) = a - ab + b^2 R_i = a(1-b) + b^2 R_i$ which in turn implies $(1 - b^2)R_i = a(1 - b)$, i.e., $(1 - b)(1 + b)R_i = a(1 - b)$, which implies $R_i = \frac{a}{1+b}$, assuming $b \neq 1$. Note that $b \geq 0$.

4 Conclusion and Open Questions

We present a generalization of the classic multiple access model, by introducing the notion of soft collisions, which enable different transmissions to succeed simultaneously. This new model captures the fact that collisions are a phenomenon experienced by the receiving end of transmissions, and it depends on the amount of interferences sensed by this receiver from the various simultaneous transmissions.

We show that for homogeneous interferences, if agents are selfish, then the system's performance at equilibrium can be up to an exponential factor far away from the optimal performance. We further introduce a penalty function to be cast on the agents, inducing a much better performance in an equilibrium, which is at most a factor of e away from the optimal performance.

Several interesting questions remain open. The main question is, of course, obtaining analytic guarantees as to the price of anarchy and the price of stability for non-homogeneous interferences. We believe that our results serve as a mere first step in understanding such interference-bound environments. Furthermore, while still considering homogeneous interferences, it would be interesting to see if there exists a penalization scheme where the penalty incurred by an agent depends solely upon his chosen strategy, which results in a sub-exponential price of anarchy, for the case of homogeneous interferences in the case where $\alpha \geq 2/e$. Another important goal is to try and use the intuition gained by these analytic results in an attempt to devise better medium-access protocols, taking into account possible prior knowledge of inter-agents interferences.

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